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On square lattice directed percolation and resistance models†

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Abstract. Recent papers by Dhar *et al* and Dhar showed that the standard directed percolation model, with blocked or one-way bonds, is dual to a percolation model with one-way or two-way bonds. This duality is exploited to improve lower bounds for the critical probability of the directed bond percolation model on the square lattice by considering k-order backflows. Two intuitive claims made in the papers mentioned above are verified in this paper. Wedge angles of the directed percolation model and the dual model exist almost surely and sum to π. As the order of backflows $k \to \infty$, the limit of the Dhar–Barma–Phani bounds is the correct critical probability. The proofs use techniques from the theories of subadditive processes and first passage percolation.

1. Introduction

1.1. DI and DR models

Consider the bond percolation model on the square lattice in which each bond is blocked with probability p_0, one-way (horizontal bonds allow passage to the right only, vertical bonds up only) with probability p_1, and two-way with probability p_2, with $p_0+p_1+p_2=1$, independently of all other bonds. The blocked, one-way and two-way bonds may be thought of as insulators, diodes and resistors, respectively. In the terminology of Dhar *et al* (1981) the standard directed percolation models (where $p_2=0$) are called diode–insulator percolation models (denoted here by DI), and their dual models (where $p_0=0$) are called diode–resistor percolation models (denoted by DR). Let DI(p) represent the model with $p_0=1-p$, $p_1=p$ and $p_2=0$ for $0 \leq p \leq 1$, and DR(p) represent the model with $p_0=0$, $p_1=1-p$ and $p_2=p$ for $0 \leq p \leq 1$. The models DI(p) and DR$(1-p)$ form a dual pair under the duality relationship of Dhar *et al* (1981) described in § 1.3.

1.2. Critical probabilities

For the DI(p) model, let $P_{\mathrm{DI}}(p)$ denote the probability that fluid from a source at the origin wets an infinite set of sites. $P_{\mathrm{DI}}(p)$ is called the percolation probability. Define the critical probability for the DI model by

$$p_{\mathrm{DI}} = \inf\{0 \leq p \leq 1 : P_{\mathrm{DI}}(p) > 0\}.$$

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By translation invariance of the models, the quantities $P_{\rm DI}(p)$ and $p_{\rm DI}$ remain unchanged if defined in terms of a fixed site other than the origin.

For the DR(p) model, if fluid wets a site at (x, y), every site (x_1, y_1) with $x_1 \geq x$ and $y_1 \geq y$ is also wetted. If (x, y) is dry, then all sites (x_2, y_2) with $x_2 \leq x$ and $y_2 \leq y$ are also dry. The set of wetted sites W, and the set of dry sites D, if both are non-empty, are separated by the set Δ of bonds which have endpoints in both W and D. Note that Δ is typically not a connected set of bonds. Δ is called the 'staircase' in Dhar *et al* (1981).

The percolation probability of the DR(p) model, denoted $P_{\rm DR}(p)$, is the probability that fluid from a single source located at the origin wets every site of the lattice. Define the critical probability

$$p_{\rm DR} = \inf\{0 \leq p \leq 1 \colon P_{\rm DR}(p) > 0\}.$$

1.3. Duality

To describe the duality between DI and DR models, consider the DR model defined on the square lattice S. The Whitney dual lattice S^* is also a square lattice with its set of bonds in one-to-one correspondence with the set of bonds of S. Denote the bond in S^* corresponding to $b \in S$ by b^*. For any bond b in S or S^*, let $X(b) = \rm D$, I or R if the bond b is a diode, insulator or resistor, respectively. Define a configuration on S^* from a DR model configuration on S by letting $X(b^*) = \rm D$, allowing passage upwards or to the left, if $X(b) = \rm D$, and letting $X(b^*) = \rm I$ if $X(b) = \rm R$. (Alternatively, the diodes in S^* could allow passage downwards or to the right.) If $\Delta \neq \varnothing$, the dual bonds corresponding to the bonds in Δ form a doubly infinite connected path in the dual lattice along which fluid could pass in the directed model. By rotation, the configuration on the dual lattice is seen to be equivalent to a configuration of the DI model. The staircase Δ corresponds to the edge of the 'backbone' of the infinite cluster in the DI model, i.e. the set of sites which are wetted and from which infinitely many sites are wetted. If $p > p_{\rm DR}$, there is no infinite cluster in the DI model, so $1 - p \leq p_{\rm DI}$. Thus, $p_{\rm DI} + p_{\rm DR} \leq 1$. If $p < p_{\rm DR}$, then $\Delta \neq \varnothing$ almost surely (AS), so there is an infinite cluster in the DI model with probability one. By countability of the sites of S^*, and translation invariance, there exists an infinite cluster in the DI model from any fixed source site with positive probability. Then $1 - p \geq p_{\rm DI}$, so we also have $p_{\rm DI} + p_{\rm DR} \geq 1$. Together, $p_{\rm DI} + p_{\rm DR} = 1$.

1.4. Wedge angle

Dhar *et al* (1981) state that angles $\theta_{\rm DI}(p)$ and $\theta_{\rm DR}(p)$ exist such that for $p > p_{\rm DI}$ the infinite cluster in the DI model is confined to a wedge of angle $\theta_{\rm DI}(p)$, for $p < p_{\rm DR}$ the wetted region in the DR model is contained in a wedge of angle $\theta_{\rm DR}(p)$, and that by the duality relationship, for $p > p_{\rm DI}$.

$$\theta_{\rm DI}(p) + \theta_{\rm DR}(1 - p) = \pi,$$

because the maximum and minimum inclinations of infinite paths in the DI model are identical to the inclinations of the boundary Δ^* of the wetted region in the dual DR model. This argument is valid if the limit $\lim_{n\to\infty} X_n/n$ exists, where $X_n = \inf\{k \in \mathbb{Z} \colon (n, k)$ is wetted by fluid from the origin in the DR model$\}$, which implies that an asymptotic direction exists for Δ and that the DR model wetted region is confined in a wedge of angle $\theta_{\rm DR}(1 - p) = 2 \tan^{-1}(-\lim_{n\to\infty} X_n/n) + \frac{1}{2}\pi$ asymptotically.

However, if $\liminf_{n\to\infty} X_n/n < \limsup_{n\to\infty} X_n/n$, any confining wedge angles $\theta_{DI}(p)$ and $\theta_{DR}(1-p)$ then satisfy $\theta_{DI}(p)+\theta_{DR}(1-p)>\pi$. In § 2, techniques from subadditive process theory, used for equating time constants in first passage percolation (Smythe and Wierman 1978), are applied to prove that $\lim_{n\to\infty}(X_n/n)$ exists and is constant with probability one.

1.5. *Convergence of lower bounds*

For $k \geq 1$, a k-order backflow path is a directed path p such that every site $P=(x_0, y_0) \in p$ is followed only by sites $(x, y) \in p$ with $x \geq x_0 - k + 1$. Dhar provides an upper bound for the critical probability of the DR(p) model by finding $\lim_{n\to\infty} Y_n^{(k)}/n$, where $Y_n^{(k)} = \inf\{k \in \mathbb{Z}_+ : (n, k)$ is wetted by fluid from the origin through a k-order backflow path in $\{(x, y): x \leq n\}\}$. Dhar states that as $k \to \infty$ the upper bounds converge to the correct critical probability. Section 3 provides a proof of this claim, using the methods of § 2.

2. Existence of wedge angle

2.1. *A subadditive process result*

We first state a special case of a result due to Hammersley (1974) which is related to the theory of subadditive processes. A proof may be found in Smythe and Wierman (1978 p 20).

Theorem 2.1. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of random variables with distribution functions F_n and finite second moments. Suppose that for each pair (m, n) of positive integers with $m < n$ there exists a random variable X'_{mn} satisfying

(i) X'_{mn} has distribution function F_n

(ii) X_m and X'_{mn} are independent

(iii) $F_{n+m} \geq F_m * F_n$ for all m and n, where $*$ denotes the convolution operator

(iv) X_n is monotone in n.

Then $E(X_n)$ is a subadditive function in n, and $X_n/n \to \gamma$ almost surely, where $\gamma = \lim E(X_n/n) = \inf E(X_n/n)$.

2.2. *The cylinder-restricted process*

The Hammersley result will be applied to a sequence of random variables that approximates the X_n sequence which defines the wedge angle in the DR(p) model. Let $S_n = \inf\{k \in \mathbb{Z}$: (n, k) is wetted from the origin through a path lying entirely in $\{(x, y): 0 \leq x < n\}$ except for the final endpoint$\}$. We will refer to $\{S_n\}$ as the cylinder-restricted process. Let F_n denote the cumulative distribution function of S_n.

Lemma 2.2 $\lim_{n\to\infty} S_n/n = \gamma_0$ AS, where $\gamma_0 = \inf E(S_n/n)$.

Proof. We verify the hypotheses of theorem 2.1. Let $S'_{mn} = \inf\{k \in \mathbb{Z}$: $(n+m, S_m+k)$ may be wetted from (m, S_m) through a path lying entirely in $\{(x, y): m \leq y < m+n\}$ except for the final endpoint$\}$. Note that the union of the cylinder path from $(0, 0)$ to (m, S_m) and the cylinder path from (m, S_m) to $(m+n, S_m+S'_{m,n})$ is a cylinder path from (0, 0) to $(m-n, S_m+S'_{m,n})$. Taking the infimum, we obtain

$$S_{m+n} \leqslant S_m + S'_{m,n}. \tag{1}$$

Note that by independence of bonds in $x<m$ and $x \geqslant m$, and translation invariance,

$$P(S'_{mn} \leqslant \alpha | S_m = k) = P(S_n \leqslant \alpha) \qquad \forall \alpha \in \mathbb{R}$$

so S'_{mn} and S_m are stochastically independent and S'_{mn} has distribution function F_n. Then by inequality (1), condition (iii) of theorem 2.1 is satisfied.

In the DR(p) model, each vertical bond is directed upward with probability p. Thus, for any integer i, the m vertical bonds in $B_{i,m} = \{(x, y): 0 \leqslant x < m, i \leqslant y \leqslant i+1\}$ are all directed upwards simultaneously with probability p^m. In this case, there is a barrier which fluid cannot pass through from top to bottom. For fixed m, the events of existence of barriers in $B_{i,m}$ are independent events. Then for $i \leqslant -1, P(S_m \leqslant i) \leqslant P(\nexists$ barriers in $B_{m,-1}, \ldots, B_{m,i}) = (1-p^m)^{-i}$, so $E(S_m)^2 < \infty$ as required for theorem 2.1. Since (n, S_n) is wetted in the DR(p) model for any $n \geqslant m$, the sequence $\{S_n\}$ is monotone non-increasing, so theorem 2.1 may be applied to prove almost sure convergence.

2.3. *Relaxation of cylinder restrictions*

Define a cylinder-restricted process, for each k, by $S_n^{(k)} = \inf\{i \in \mathbb{Z}: (n, i)$ is wetted from the origin through a path lying entirely in $\{(x, y): -k \leqslant x < n+k\}\}$.

Lemma 2.3. $\lim_{n\to\infty} S_n^{(k)}/n = \gamma_0$ AS and $\lim_{n\to\infty} E[S_n^{(k)}/n] = \inf E[S_n^{(k)}/n] = \gamma_0$.

Proof. Define a shifted version of S_n by $T_n^{(k)} = \inf\{i \in \mathbb{Z}: (n-k, i)$ is wetted from $(-k, 0)$ through a path lying entirely in $\{(x, y): k \leqslant x < n-k\}$ except for the final endpoint$\}$. Note that $T_n^{(k)}$ and S_n are identically distributed, so that for each fixed k, $\lim_{n\to\infty} E[T_n^{(k)}/n] = \gamma_0$. Since the union of the line segment from $(-k, 0)$ to $(0, 0)$, the $-k \leqslant x < n+k$ cylinder path from $(0, 0)$ to $(n, S_n^{(k)})$ and the line segment from $(n, S_n^{(k)})$ to $(n+k, S_n^{(k)})$ gives $a -k \leqslant x < n+k$ cylinder path, we have

$$T_{n+2k}^{(k)} \leqslant S_n^{(k)} \leqslant S_n \qquad \forall n, k \geqslant 1. \tag{2}$$

Applying lemma 2.2, we have that

$$\lim_{n\to\infty} S_n/n = \gamma_0 \text{ AS} \qquad \text{and } \lim_{n\to\infty} T_n^{(k)}/n = \gamma_0 \text{ AS}.$$

Thus, $\lim_{n\to\infty} T_{n+2k}^{(k)}/n = (\lim_{n\to\infty} T_{n+2k}^{(k)}/(n+2k))(\lim_{n\to\infty}(n+2k)/n) = \gamma_0$ AS also, which implies that $\lim_{n\to\infty} S_n^{(k)}/n = \gamma_0$ AS for each fixed k. Taking expectations in (2),

$$E[T_{n+2k}^{(k)}] \leqslant E[S_n^{(k)}] \leqslant E[S_n],$$

which, using $\lim_{n\to\infty} E[T_n^{(k)}/n] = \lim_{n\to\infty} E[S_n/n] = \gamma_0$, implies that $\lim_{n\to\infty} E[S_n^{(k)}/n] = \gamma_0$ for each fixed k. Since $E[S_n^{(k)}]$ is subadditive in n for each fixed k, we have that $E[S_n^{(k)}/n] \geqslant \gamma_0$ for all $n, k \geqslant 1$, so $\inf E[S_n^{(k)}/n] = \gamma_0$.

2.4. *Convergence of expectations*

We now begin consideration of the process of interest, $\{X_n\}$.

Lemma 2.4. $\lim_{n\to\infty} E[X_n/n] = \gamma_0$.

Proof. Note that for all positive integers n and k,

$$X_n \leqslant S_n^{(k+1)} \leqslant S_n^{(k)} \leqslant S_n.$$

For n and ω fixed, $S_n^{(k)}(\omega)$ is non-increasing in k, and $\lim_{k\to\infty} S_n^{(k)}(\omega) = X_n(\omega)$. By the monotone convergence theorem,

$$\lim_{k\to\infty} E[S_n^{(k)}/n] = E[X_n/n].$$

Since $E[S_n^{(k)}/n] \geqslant \gamma_0$ for all n and k by lemma 2.3, we have $E[X_n/n] \geqslant \gamma_0$ for all n, so

$$\gamma_0 \leqslant \lim_{n\to\infty} E[X_n/n] \leqslant \lim_{n\to\infty} E[S_n/n] = \gamma_0.$$

2.5. *Almost sure convergence*

Theorem 2.5. $\lim_{n\to\infty} X_n/n = \gamma_0$ AS.

Proof. Let $Y = \inf\{i \in \mathbb{Z}$: $(0, i)$ is wetted by fluid from the origin$\}$. We consider two cases.

Case 1. Suppose $E[Y] = -\infty$. Then

$$E[X_n] \leqslant E[Y] = -\infty \qquad \forall n \in \mathbb{Z}_+,$$

so $\gamma_0 = -\infty$ by lemma 2.4. Hence

$$\limsup_{n\to\infty} X_n/n \leqslant \lim_{n\to\infty} S_n/n = \gamma_0 = -\infty \text{ AS}.$$

Case 2. Suppose $E[Y] > -\infty$. Define shifted versions of Y by $Y(x, y) = \inf\{i \in \mathbb{Z}$: $(x, y+i)$ is wetted by fluid from $(x, y)\}$, and shifted versions of S_n by $S_n(x_0, y_0) = \inf\{i \in \mathbb{Z}$: (x_0+n, i) is wetted from (x_0, y_0) through a path lying entirely in $\{(x, y)$: $x_0 \leqslant x < x_0 + n\}$ except for the final endpoint$\}$. Define processes restricted to the right half-plane by $R_n(x_0, y_0) = \inf\{i \in \mathbb{Z}$: (x_0+n, y_0+i) is wetted from (x_0, y_0) through paths contained entirely in $x \geqslant x_0\}\}$. For convenience, let $R_n(0, 0) = R_n$.

Lemma 2.5. $\lim_{n\to\infty} R_n/n = \gamma_0$ AS.

Proof. Note that

$$S_n + Y(n, S_n - 1) \leqslant R_n \leqslant S_n \tag{3}$$

since any path to a site (n, y) with $y < S_n + Y(n, S_n - 1)$ must lie above and to the right of the lowest paths for S_n and $Y(n, S_n - 1)$ for $y \geqslant S_n + Y(n, S_n - 1)$, and thus is wetted by fluid flowing to the right from them.

Also, since on the set $\{S_n = j\}$, $Y(n, S_n - 1)$ depends only on the bonds in the region $y < j - 1$, and S_n depends only on bonds in $y \geqslant j - 1$, we have

$$\begin{aligned} P[Y(n, S_n - 1) = i] &= \sum_{j=-\infty}^{0} P[Y(n, S_n - 1) = i, S_n = j] \\ &= \sum_{j=-\infty}^{0} P[Y(n, S_n - 1) = i | S_n = j] P[S_n = j] \\ &= \sum_{j=-\infty}^{0} P[Y = i] P[S_n = j] = P[Y = i]. \end{aligned}$$

Therefore, the distribution of $Y(n, S_n - 1)$ is independent of S_n, and in particular $E[Y(n, S_n - 1)] = E[Y]$ for all n.

Then $E|Y| < \infty$ implies that $\Sigma_{n=1}^{\infty} P[Y(n, S_n - 1)/n > \varepsilon] < \infty$ for every $\varepsilon > 0$. By the Borel–Cantelli lemma,

$$\lim_{n\to\infty} Y(n, S_n - 1)/n = 0 \text{ AS.}$$

By (3) and lemma 2.2, we have

$$\liminf R_n/n \geqslant \gamma_0 \text{ AS}$$

so in fact $\lim_{n\to\infty}(R_n/n) = \gamma_0$ AS.

To complete the proof of case 2, we first note that since the events $B_i = \{\omega \in \Omega: \lim_{n\to\infty} R_n(i, 0) = \gamma_0\}$ each have probability one, we have $P[\bigcap_{i=-\infty}^{0} B_i] = 1$ and thus $P[\bigcap_{i=-\infty}^{0} B_i \cap \{Y < \infty\}] = 1$. For $\omega \in \bigcap_{i=-\infty}^{0} B_i \cap \{Y < \infty\}$

$$Y(\omega) + R_n(0, Y)(\omega) \leqslant X_n(\omega)$$

implies that

$$\liminf_{n\to\infty} \frac{X_n(\omega)}{n} \geqslant \lim_{n\to\infty} \frac{Y(\omega)}{n} + \lim_{n\to\infty} \frac{R_n(0, Y)(\omega)}{n} = 0 + \gamma_0 = \gamma_0.$$

We conclude that

$$\liminf_{n\to\infty} X_n/n \geqslant \gamma_0 \text{ AS.}$$

Since we trivially have

$$\limsup_{n\to\infty} X_n/n \leqslant \limsup_{n\to\infty} S_n/n \leqslant \gamma_0 \text{ AS}$$

the theorem is proved.

3. Convergence of bounds to the critical probability

Suppose $E[Y] > -\infty$. Then there is a finite constant γ_0 such that $\lim_{n\to\infty} X_n/n = \gamma_0$ AS. Since $\{X_n\}$ is a non-increasing sequence, then $P[X_n > -\infty$ for all $n] = 1$. For $\omega \in \{X_n > -\infty$ for all $n\}$, for each n there exists $k(\omega)$ such that there is a $k(\omega)$-order backflow path from the origin to $(n, X_n(\omega))$, so $Y_n^{(k)}(\omega) = X_n(\omega)$ for all $k \geqslant k(\omega)$. Thus, for each fixed n, $Y_n^{(k)}$ is a non-increasing sequence in k, and $Y_n^{(k)} \to X_n$ almost surely as $k \to \infty$. By the monotone convergence theorem,

$$\lim_{k\to\infty} E[Y_n(k)/n] = E[X_n/n]. \tag{4}$$

Define shifted versions of $Y_n^{(k)}$ by $Y_n^{(k)}(x_0, y_0) = \inf\{i \in \mathbb{Z}: (x_0 + n, y_0 + i)$ is wetted by fluid from the origin through a k-order backflow path in the region $x_0 \leqslant x \leqslant x + n\}$. Notice that

$$Y_{m+n}^{(k)} \leqslant Y_m^{(k)} + Y_n^{(k)}(m, Y_m^{(k)}).$$

An argument similar to that in lemma 2.5 shows that

$$E[Y_n^{(k)}(m, Y_m^{(k)})] = E[Y_n^{(k)}] \qquad \forall n, k.$$

Therefore,

$$E[Y^{(k)}_{m+n}] \leqslant E[Y^{(k)}_m] + E[Y^{(k)}_n],$$

so for each fixed k, $E[Y^{(k)}_n]$ is a subadditive function of n. A standard result in the theory of subadditive functions (Hille and Phillips 1957 p 244) then implies that

$$\lim_{n\to\infty} E[Y^{(k)}_n/n] = \inf_{n\to\infty} E[Y^{(k)}_n/n] = \gamma_k \text{ exists.}$$

Choose $\varepsilon < 0$. There exists n_0 sufficiently large that $E[X_{n_0}/n_0] < \gamma_0 + \varepsilon$. By (4) there exists a k_0 such that $E[Y^{(k_0)}_{n_0}/n_0] < \gamma_0 + 2\varepsilon$, which implies that

$$\gamma_{k_0} = \inf_{n\to\infty} E[Y^{(k_0)}_n/n] < \gamma_0 + 2\varepsilon.$$

Since γ_k is non-increasing in k, and $\varepsilon < 0$ is arbitrary,

$$\lim_{k\to\infty} \gamma_k \leqslant \gamma_0.$$

Because $X_n \leqslant Y^{(k)}_n$ for all k and n, by lemma 2.4 we also have

$$\gamma_0 \leqslant \lim_{k\to\infty} \gamma_k.$$

For $k = 0, 1, 2, \ldots$, let $\gamma_k \equiv \gamma_k(p)$ to indicate explicitly the dependence on the parameter p of the DR(p) model and define $p_{\rm DR}(k) = \sup\{p \in [0, 1]: \gamma_k(p) \geqslant -1\}$.

As indicated by Dhar *et al*, a slope $\gamma_0(p) < -1$ implies that an infinite cluster exists in the DI($1-p$) model with probability zero. Then the entire lattice is wetted from the origin in the DR(p) model with probability one, so in fact $E[Y] = -\infty$ and $\gamma_0(p) = -\infty$. Thus, $p < p_{\rm DR}(0)$ implies that $p < p_{\rm DR}$, so $p_{\rm DR}(0) \leqslant p_{\rm DR}$. Trivially $p > p_{\rm DR}(0)$ implies that $p > p_{\rm DR}$, so in fact $p_{\rm DR}(0) = p_{\rm DR}$, the correct critical probability.

Since $\gamma_0(p) \leqslant \gamma_{k+1}(p)$ for all $p \in [0, 1]$, we have $p_{\rm DR}(0) \leqslant p_{\rm DR}(k+1) \leqslant p_{\rm DR}(k)$ for all k, so $p_{\rm DR}(0) \leqslant \lim_{k\to\infty} p_{\rm DR}(k)$. On the other hand, if $p > p_{\rm DR}(0)$, then $\gamma_0(p) < -1$, so $\gamma_k(p) < -1$ for some k, implying that $p > \lim_{k\to\infty} p_{\rm DR}(k)$. Thus, $p_{\rm DR}(0) \geqslant \lim_{k\to\infty} p_{\rm DR}(k)$ also, so

$$p_{\rm DR} = p_{\rm DR}(0) = \lim_{k\to\infty} p_{\rm DR}(k),$$

so Dhar's bounds converge to the correct critical probability.

References

Dhar D 1982 *J. Phys. A: Math. Gen.* **15** 1849–58
Dhar D, Barma M and Phani M K 1981 *Phys. Rev. Lett.* **47** 1238–241
Gray L, Smythe R T and Wierman J C 1980 *J. Appl. Prob.* **17** 979–86
Hammersley J M 1974 *Ann. Prob.* **2** 652–80
Hille E and Phillips R S 1957 *Functional Analysis and Semi-Groups* (Providence, RI, USA: Am. Math. Soc.)
Mauldon J G 1961 *Proc. 4th Berkley Symp Math. Stat.* **2** 337–45
Smythe R T and Wierman J C 1978 *First-Passage Percolation on the Square Lattice* (Berlin: Springer)